

1

Statistics of Galaxy Clustering

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Abstract

In this introductory talk we will establish connections between the statistical analysis of galaxy clustering in cosmology and recent work in mainstream spatial statistics. The lecture will review the methods of spatial statistics used by both sets of scholars, having in mind the cross-fertilizing purpose of the meeting series. Special topics will be: description of the galaxy samples, selection effects and biases, correlation functions, nearest neighbor distances, void probability functions, Fourier analysis, and structure statistics.

1.1 Introduction

One of the most important motivations of these series of conferences is to promote vigorous interaction between statisticians and astronomers. The organizers merit our admiration for bringing together such a stellar cast of colleagues from both fields. In this third edition, one of the central subjects is cosmology, and in particular, statistical analysis of the large-scale structure in the universe. There is a reason for that — the rapid increase of the amount and quality of the available observational data on the galaxy distribution (also on clusters of galaxies and quasars) and on the temperature fluctuations of the microwave background radiation.

These are the two fossils of the early universe on which cosmology, a science driven by observations, relies. Here we will focus on one of them — the galaxy distribution. First we briefly review the redshift surveys, how they

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are built and how to extract statistically analyzable samples from them, considering selection effects and biases. Most of the statistical analysis of the galaxy distribution are based on second order methods (correlation functions and power spectra). We comment them, providing the connection between statistics and estimators used in cosmology and in spatial statistics. Special attention is devoted to the analysis of clustering in Fourier space, with new techniques for estimating the power spectrum, which are becoming increasingly popular in cosmology. We show also the results of applying these second-order methods to recent galaxy redshift surveys.

Fractal analysis has become very popular as a consequence of the scale-invariance of the galaxy distribution at small scales, reflected in the power-law shape of the two-point correlation function. We discuss here some of these methods and the results of their application to the observations, supporting a gradual transition from a small-scale fractal regime to large-scale homogeneity. The concept of lacunarity is illustrated with some detail.

We end by briefly reviewing some of the alternative measures of point statistics and structure functions applied thus far to the galaxy distribution: void probability functions, counts-in-cells, nearest neighbor distances, genus, and Minkowski functionals.

1.2 Cosmological datasets

Cosmological datasets differ in several respects from those usually studied in spatial statistics. The point sets in cosmology (galaxy and cluster surveys) bear the imprint of the observational methods used to obtain them.

The main difference is the systematically variable intensity (mean density) of cosmological surveys. These surveys are usually magnitude-limited, meaning that all objects, which are brighter than a pre-determined limit, are observed in a selected region of the sky. This limit is mainly determined by the telescope and other instruments used for the program. Apparent magnitude, used to describe the limit, is a logarithmic measure of the observed radiation flux.

It is usually assumed that galaxies at all distances have the same (universal) luminosity distribution function. This assumption has been tested and found to be in satisfying accordance with observations. As the observed flux from a galaxy is inversely proportional to the square of its distance, we can see at larger distances only a bright fraction of all galaxies. This leads directly to the mean density of galaxies that depends on their distance from us r .

This behaviour is quantified by a selection function $\phi(r)$, which is usually found by estimating first the luminosity distribution of galaxies (the luminosity function).

One can also select a distance limit, find the minimum luminosity of a galaxy, which can yet be seen at that distance, and ignore all galaxies that are less luminous. Such samples are called volume-limited. They are used for some special studies (typically for counts-in-cells), but the loss of hard-earned information is enormous. The number of galaxies in volume-limited samples is several times smaller than in the parent magnitude-limited samples. This will also increase the shot (discreteness) noise.

In addition to the radial selection function $\phi(r)$, galaxy samples also are frequently subject to angular selection. This is due to our position in the Galaxy — we are located in a dusty plane of the Galaxy, and the window in which we see the Universe, also is dusty. This dust absorbs part of galaxies' light, and makes the real brightness limit of a survey dependent on the amount of dust in a particular line-of-sight. This effect has been described by a $\phi(b) \sim (\sin b)^{-1}$ law (b is the galactic latitude); in reality the dust absorption in the Galaxy is rather inhomogeneous. There are good maps of the amount of Galactic dust in the sky, the latest maps have been obtained using the COBE and IRAS satellite data (Schlegel et al. 1998).

Edge problems, which usually affect estimators in spatial statistics, also are different for cosmological samples. The decrease of the mean density towards the sample borders alleviates these problems. Of course, if we select a volume-limited sample, we select also all these troubles (and larger shot noise). From the other side, edge effects are made more prominent by the usual observing strategies, when surveys are conducted in well-defined regions in the sky. Thus, edge problems are only partly alleviated; maybe it will pay to taper our samples at the side borders, too?

Some of the cosmological surveys have naturally soft borders. These are the all-sky surveys; the best known is the IRAS infrared survey, dust is almost transparent in infrared light. The corresponding redshift survey is the PSCz survey, which covers about 85% of the sky (Saunders et al. 2000). A special follow-up survey is in progress to fill in the remaining Galactic Zone-of-Avoidance region, and meanwhile numerical methods have been developed to interpolate the structures seen in the survey into the gap (Schmoldt et al. 1999, Saunders & Ballinger 2000).

Another peculiarity of galaxy surveys is that we can measure exactly only the direction to the galaxy (its position in the sky), but not its distance. We measure the radial velocity v_r (or redshift $z = v_r/c$, c is the velocity of light) of a galaxy, which is a sum of the Hubble expansion, proportional to the distance d , and the dynamical velocity v_p of the galaxy, $v_r = H_0 d + v_p$. Thus we are differentiating between redshift space, if the distances simply are determined as $d = v_r/H_0$, and real space. The real space positions of galaxies could be calculated if we exactly knew the peculiar velocities of galaxies; we do not. The velocity distortions can be severe; well-known features of redshift space are fingers-of-God, elongated structures that are caused by a large radial velocity dispersion in massive clusters of galaxies.

The velocity distortions expand a cluster in redshift space in the radial direction five-ten times.

For large-scale structures the situation is different, redshift distortions compress them. This is due to the continuing gravitational growth of structures. These differences can best be seen by comparing the results of numerical simulations, where we know also the real-space situation, in redshift space and in real space.

The last specific feature of the cosmology datasets is their size. Up to recent years most of the datasets have been rather small, of the order of 10^3 objects; exceptions exist, but these are recent. Such a small number of points gives a very sparse coverage of three-dimensional survey volumes, and shot noise has been a severe problem.

This situation is about to change, swinging to the other extreme; the membership of new redshift surveys already is measured in terms of 10^5 (160,000 for the 2dF survey, quarter of a million planned) and million-galaxy surveys are on their way (the Sloan Survey). More information about these surveys can be found in their Web pages: <http://www.mso.anu.edu.au/2dFGRS/> for the 2dF survey and <http://www.sdss.org/> for the Sloan survey. This huge amount of data will force us to change the statistical methods we use. Nevertheless, the deepest surveys (e.g., distant galaxy cluster surveys) will always be sparse, so discovering small signals from shot-noise dominated data will remain a necessary art.

1.3 Correlation analysis

There are several related quantities that are second-order characteristics used to quantify clustering of the galaxy distribution in real or redshift space. The most popular one in cosmology is the two-point correlation function, $\xi(\mathbf{r})$. The infinitesimal interpretation of this quantity reads as follows:

$$dP_{12} = \bar{n}^2 [1 + \xi(\mathbf{r})] dV_1 dV_2 \quad (1.1)$$

is the joint probability that in each one of the two infinitesimal volumes dV_1 and dV_2 , with separation vector \mathbf{r} , lies a galaxy. Here \bar{n} is the mean number density (intensity). Assuming that the galaxy distribution is a homogeneous (invariant under translations) and isotropic (invariant under rotations) point process, this probability depends only on $r = |\mathbf{r}|$. In spatial statistics, other functions related with $\xi(r)$ are commonly used:

$$\lambda_2(r) = \bar{n}^2 \xi(r) + 1, \quad g(r) = 1 + \xi(r), \quad \Gamma(r) = \bar{n}(\xi(r) + 1), \quad (1.2)$$

where $\lambda_2(r)$ is the second-order intensity function, $g(r)$ is the pair correlation function, also called the radial distribution function or structure function, and $\Gamma(r)$ is the conditional density proposed by Pietronero (1987).

Different estimators of $\xi(r)$ have been proposed so far in the literature, both in cosmology and in spatial statistics. The main differences are in correction for edge effects. Comparison of their performance can be found in several papers (Pons-Bordería et al. 1999, Kerscher et al. 2000, Stoyan & Stoyan 2000). There is clear evidence that $\xi(r)$ is well described by a power-law at scales $0.1 \leq r \leq 10 h^{-1}$ Mpc where h is the Hubble constant in units of $100 \text{ km s}^{-1} \text{ Mpc}^{-1}$:

$$\xi(r) = \left(\frac{r}{r_0} \right)^{-\gamma},$$

with $\gamma \simeq 1.8$ and $r_0 \simeq 5.4 h^{-1}$ Mpc. This scaling behavior is one of the reasons that have lead some astronomers to describe the galaxy distribution as fractal. A power-law fit for $g(r) \propto r^{3-D_2}$ permits to define the correlation dimension D_2 . The extent of the fractal regime is still a matter of debate in cosmology, but it seems clear that the available data on redshift surveys indicate a gradual transition to homogeneity for scales larger than $15\text{--}20 h^{-1}$ Mpc (Martínez 1999). Moreover, in a fractal point distribution, the correlation length r_0 increases with the radius of the sample because the mean density decreases (Pietronero 1987). This simple prediction of the fractal interpretation is not supported by the data, instead r_0 remains constant for volume-limited samples with increasing depth (Martínez et al. 2001).

Several versions of the volume integral of the correlation function are also frequently used in the analysis of galaxy clustering. The most extended one in spatial statistics is the so-called Ripley K -function

$$K(r) = \int_0^r 4\pi s^2 (1 + \xi(s)) ds \quad (1.3)$$

although in cosmology it is more frequent to use an expression which provides directly the average number of neighbors an arbitrarily chosen galaxy has within a distance r , $N(< r) = \bar{n}K(r)$ or the average conditional density

$$\Gamma^*(r) = \frac{3}{r^3} \int_0^r \Gamma(s) s^2 ds$$

Again a whole collection of estimators are used to properly evaluate these quantities. Pietronero and coworkers recommend to use only minus-estimators to avoid any assumption regarding the homogeneity of the process. In these estimators, averages of the number of neighbors within a given distance are taken only considering as centers these galaxies whose distances to the border are larger than r . However, caution has to be exercised with this procedure, because at large scales only a small number of centers remain, and thus the variance of the estimator increases.

Integral quantities are less noisy than the corresponding differential expressions, but obviously they do contain less information on the clustering process due the fact that values of $K(r_1)$ and $K(r_2)$ for two different scales

r_1 and r_2 are more strongly correlated than values of $\xi(r_1)$ and $\xi(r_2)$. Scaling of $N(< r) \propto r^{D_2}$ provides a smoother estimation of the correlation dimension. If scaling is detected for partition sums defined by the moments of order q of the number of neighbors

$$Z(q, r) = \frac{1}{N} \sum_{i=1}^N n_i(r)^{q-1} \propto r^{D_q/(q-1)},$$

the exponents D_q are the so-called generalized or multifractal dimensions (Martínez et al. 1990). Note that for $q = 2$, $Z(2, r)$ is an estimator of $N(< r)$ and therefore D_q for $q = 2$ is simply the correlation dimension. If different kinds of cosmic objects are identified as peaks of the continuous matter density field at different thresholds, we can study the correlation dimension associated to each kind of object. The multiscaling approach (Jensen et al. 1991) associated to the multifractal formalism provides a unified framework to analyze this variation. It has been shown (Martínez et al. 1995) that the value of D_2 corresponding to rich galaxy clusters (high peaks of the density field) is smaller than the value corresponding to galaxies (within the same scale range) as prescribed in the multiscaling approach.

Finally we want to consider the role of lacunarity in the description of the galaxy clustering (Martínez & Saar 2001). In Fig. 1.1, we show the space distribution of galaxies within one slice of the Las Campanas redshift survey, together with a fractal pattern generated by means of a Rayleigh-Lévy flight (Mandelbrot 1982). Both have the same mass-radius dimension, defined as the exponent of the power-law that fits the variation of mass within concentric spheres centered at the observer position.

$$M(R) = FR^{D_M}. \quad (1.4)$$

The best fitted value for both point distributions is $D_M \simeq 1.6$ as shown in the left bottom panel of Fig. 1.1. The different appearance of both point distributions is a consequence of the different degree of lacunarity. Blumenfeld & Mandelbrot (1997) have proposed to quantify this effect by measuring the variability of the prefactor F in Eq. 1.4,

$$\Phi = \frac{E\{(F - \bar{F})^2\}}{\bar{F}^2}$$

The result of applying this lacunarity measure is shown in the right bottom panel of Fig. 1.1. The visual differences between the point distributions are now well reflected in this curve.

1.4 Power spectra

The current statistical model for the main cosmological fields (density, velocity, gravitational potential) is the Gaussian random field. This field is

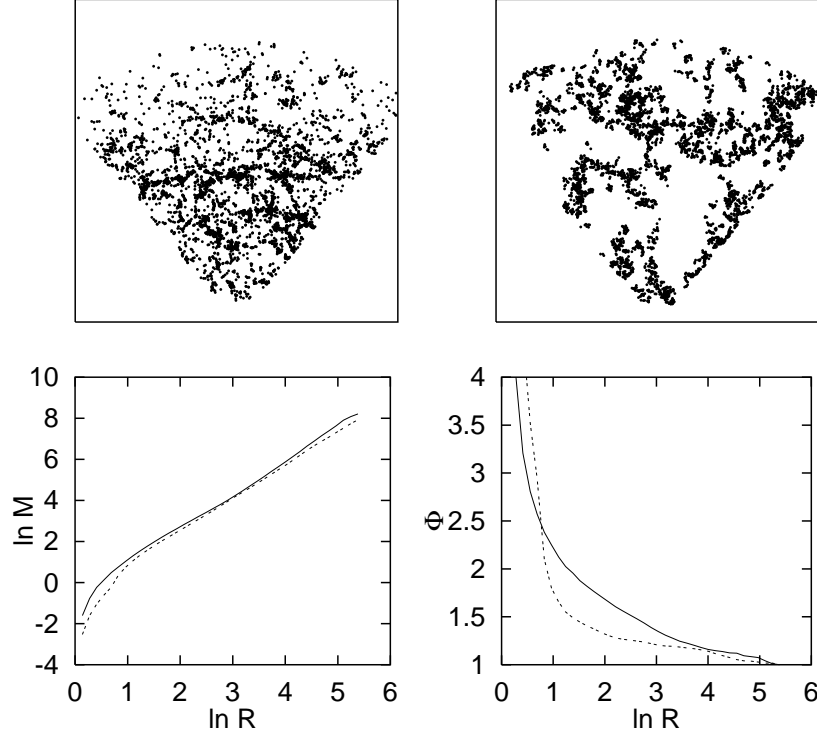


Figure 1.1. Comparison of a Las Campanas survey slice (upper left panel) with the Rayleigh-Lévy flight model (upper right panel). The fractal dimensions of both distributions coincide, as shown by the $\ln M$ – $\ln R$ curves in the lower left panel, but the lacunarity curves (in the lower right panel) differ considerably. The solid lines describe the galaxy distribution, dotted lines – the model results. From Martínez & Saar (2001).

determined either by its correlation function or by its spectral density, and one of the main goals of spatial statistics in cosmology is to estimate those two functions.

In recent years the power spectrum has attracted more attention than the correlation function. There are at least two reasons for that — the power spectrum is more intuitive physically, separating processes on different scales, and the model predictions are made in terms of power spectra. Statistically, the advantage is that the power spectrum amplitudes for different wavenumbers are statistically orthogonal:

$$E \left\{ \tilde{\delta}(\mathbf{k}) \tilde{\delta}^*(\mathbf{k}') \right\} = (2\pi)^3 \delta_D(\mathbf{k} - \mathbf{k}') P(\mathbf{k}).$$

Here $\tilde{\delta}(\mathbf{k})$ is the Fourier amplitude of the overdensity field $\delta = (\rho - \bar{\rho})/\bar{\rho}$ at a wavenumber \mathbf{k} , ρ is the matter density, a star denotes complex conjugation,

$E\{\}$ denotes expectation values over realizations of the random field, and $\delta_D(\mathbf{x})$ is the three-dimensional Dirac delta function. The power spectrum $P(\mathbf{k})$ is the Fourier transform of the correlation function $\xi(\mathbf{r})$ of the field.

Estimation of power spectra from observations is a rather difficult task. Up to now the problem has been in the scarcity of data; in the near future there will be the opposite problem of managing huge data sets. The development of statistical techniques here has been motivated largely by the analysis of CMB power spectra, where better data were obtained first, and has been parallel to that recently.

The first methods developed to estimate the power spectra were direct methods — a suitable statistic was chosen and determined from observations. A good reference is Feldman et al. (1994).

The observed samples can be modeled by an inhomogeneous point process (a Gaussian Cox process) of number density $n(\mathbf{x})$:

$$n(\mathbf{x}) = \sum_i \delta_D(\mathbf{x} - \mathbf{x}_i),$$

where $\delta_D(\mathbf{x})$ is the Dirac delta-function. As galaxy samples frequently have systematic density trends caused by selection effects, we have to write the estimator of the density contrast in a sample as

$$D(\mathbf{x}) = \sum_i \frac{\delta_D(\mathbf{x} - \mathbf{x}_i)}{\bar{n}(\mathbf{x}_i)} - 1,$$

where $\bar{n}(\mathbf{x}) \sim \bar{\rho}(\mathbf{x})$ is the selection function expressed in the number density of objects.

The estimator for a Fourier amplitude (for a finite set of frequencies \mathbf{k}_i) is

$$F(\mathbf{k}_i) = \sum_j \frac{\psi(\mathbf{x}_j)}{\bar{n}(\mathbf{x}_j)} e^{i\mathbf{k}_i \cdot \mathbf{x}} - \tilde{\psi}(\mathbf{k}_i),$$

where $\psi(\mathbf{x})$ is a weight function that can be selected at will. The raw estimator for the spectrum is

$$P_R(\mathbf{k}_i) = F(\mathbf{k}_i)F^*(\mathbf{k}_i),$$

and its expectation value

$$E\{|F(\mathbf{k}_i)|^2\} = \int G(\mathbf{k}_i - \mathbf{k}') P(\mathbf{k}') \frac{d^3 k'}{(2\pi)^3} + \int_V \frac{\psi^2(\mathbf{x})}{\bar{n}(\mathbf{x})} d^3 x,$$

where $G(\mathbf{k}) = |\tilde{\psi}(\mathbf{k})|^2$ is the window function that also depends on the geometry of the sample volume. Symbolically, we can get the estimate of the power spectra \hat{P} by inverting the integral equation

$$G \otimes \hat{P} = P_R - N,$$

where \otimes denotes convolution, P_R is the raw estimate of power, and N is the (constant) shot noise term.

In general, we have to deconvolve the noise-corrected raw power to get the estimate of the power spectrum. This introduces correlations in the estimated amplitudes, so these are not statistically orthogonal any more. A sample of a characteristic spatial size L creates a window function of width of $\Delta k \approx 1/L$, correlating estimates of spectra at that wavenumber interval.

As the cosmological spectra are usually assumed to be isotropic, the standard method to estimate the spectrum involves an additional step of averaging the estimates $\hat{P}(\mathbf{k})$ over a spherical shell $k \in [k_i, k_{i+1}]$ of thickness $k_{i+1} - k_i > \Delta k = 1/L$ in wavenumber space. The minimum-variance requirement gives the FKP (Feldman et al. 1994) weight function:

$$\psi(\mathbf{x}) \sim \frac{\bar{n}(\mathbf{x})}{1 + \bar{n}(\mathbf{x})P(k)},$$

and the variance is

$$\frac{\sigma_P^2(k)}{P_R^2(k)} \approx \frac{2}{\mathcal{N}},$$

where \mathcal{N} is the number of coherence volumes in the shell. The number of independent volumes is twice as small (the density field is real). The coherence volume is $V_c(k) \approx (\Delta k)^3 \approx 1/L^3 \approx 1/V$.

As the data sets get large, straight application of direct methods (especially the error analysis) becomes difficult. There are different recipes that have been developed with the future data sets in mind. A good review of these methods is given in Tegmark et al. (1998).

The deeper the galaxy sample, the smaller the coherence volume, the larger the spectral resolution and the larger the wavenumber interval where the power spectrum can be estimated. The deepest redshift surveys presently available are the PSCz galaxy redshift survey (15411 redshifts up to about $400h^{-1}$ Mpc, see Saunders et al. (2000)), the Abell/ACO rich galaxy cluster survey, 637 redshifts up to about $300h^{-1}$ Mpc (Miller & Batuski 2001)), and the ongoing 2dF galaxy redshift survey (141400 redshifts up to $750h^{-1}$ Mpc (Peacock et al. 2001)). The estimates of power spectra for the two latter samples have been obtained by the direct method (Miller et al. 2001, Percival et al. 2001). Fig. 1.2 shows the power spectrum for the 2dF survey.

The covariance matrix of the power spectrum estimates in Fig. 1.2 was found from simulations of a matching Gaussian Cox process in the sample volume. The main new feature in the spectra, obtained for the new deep samples, is the emergence of details (wiggles) in the power spectrum. While sometime ago the main problem was to estimate the mean behaviour of the spectrum and to find its maximum, now the data enables us to see and study the details of the spectrum. These details have been interpreted as traces of acoustic oscillations in the post-recombination power spectrum. Similar oscillations are predicted for the cosmic microwave background radiation

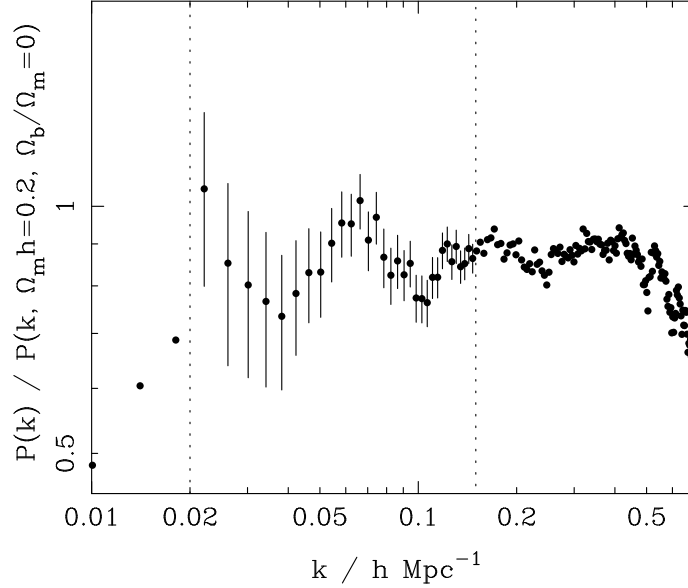


Figure 1.2. Power spectrum of the 2dF redshift survey, divided by a smooth model power spectrum. The spectrum is not deconvolved. Error bars are determined from Gaussian realizations; the dotted lines show the wavenumber region that is free of the influence of the window function and of the radial velocity distortions and nonlinear effects. (Courtesy of W. J. Percival and the 2dF galaxy redshift survey team.)

fluctuation spectrum. The CMB wiggles match the theory rather well, but the galaxy wiggles do not, yet.

Thus, the measurement of the power spectrum of the galaxy distribution is passing from the determination of its overall behaviour to the discovery and interpretation of spectral details.

1.5 Other clustering measures

To end this review we briefly mention other measures used to describe the galaxy distribution.

1.5.1 Counts-in-cells and void probability function

The probability that a randomly placed sphere of radius r contains exactly N galaxies is denoted by $P(N, r)$. In particular, for $N = 0$, $P(0, r)$ is the so-called void probability function, related with the empty space function or contact distribution function $F(r)$, more frequently used in the field of

spatial statistics, by $F(r) = 1 - P(0, r)$. The moments of the counts-in-cells probabilities can be related both with the multifractal analysis (Borgani 1993) and with the higher order n -point correlation functions (White 1979, Stoyan et al. 1995, Szapudi et al. 1999).

1.5.2 *Nearest-neighbor distributions*

In spatial statistics, different quantities based on distances to nearest neighbors have been introduced to describe the statistical properties of point processes. $G(r)$ is the distribution function of the distance r of a given point to its nearest neighbor. It is interesting to note that $F(r)$ is just the distribution function of the distance r from an arbitrarily chosen point in \mathbb{R}^3 — not being an event of the point process — to a point of the point process (a galaxy in the sample in our case). The quotient

$$J(r) = \frac{1 - G(r)}{1 - F(r)}$$

introduced by van Lieshout & Baddeley (1996) is a powerful tool to analyze point patterns and has discriminative power to compare the results of N -body models for structure formation with the real distribution of galaxies (Kerscher et al. 1999).

1.5.3 *Topology*

One very popular tool for analysis of the galaxy distribution is the genus of the isodensity surfaces. To define this quantity, the point process is smoothed to obtain a continuous density field, the intensity function, by means of a kernel estimator for a given bandwidth. Then we consider the fraction of the volume f which encompasses those regions having density exceeding a given threshold ρ_t . The boundary of these regions specifies an isodensity surface. The genus $G(S)$ of a surface S is basically the number of holes minus the number of isolated regions plus 1. The genus curve shows the variation of $G(S)$ with f or ρ_t for a given window radius of the kernel function. An analytical expression for this curve is known for Gaussian density fields. It seems that the empirical curve calculated from the galaxy catalogs can be reasonably well fitted to a Gaussian genus curve (Canavezes et al. 1998) for window radii varying within a large range of scales.

1.5.4 *Minkowski functionals*

A very elegant generalization of the previous analysis to a larger family of morphological characteristics of the point processes is provided by the Minkowski functionals. These scalar quantities are useful to study the shape and connectivity of a union of convex bodies. They are well known in spatial

statistics and have been introduced in cosmology by Mecke et al. (1994). On a clustered point process, Minkowski functionals are calculated by generalizing the Boolean grain model into the so-called germ-grain model. This coverage process consists in considering the sets $A_r = \cup_{i=1}^N B_r(\mathbf{x}_i)$ for the diagnostic parameter r , where $\{\mathbf{x}_i\}_{i=1}^N$ represents the galaxy positions and $B_r(\mathbf{x}_i)$ is a ball of radius r centered at point \mathbf{x}_i . Minkowski functionals are applied to sets A_r when r varies. In \mathbb{R}^3 there are four functionals: the volume V , the surface area A , the integral mean curvature H , and the Euler-Poincaré characteristic χ , related with the genus of the boundary of A_r by $\chi = 1 - G$. Application of Minkowski functionals to the galaxy cluster distribution can be found in Kerscher et al. (1997). These quantities have been used also as efficient shape finders by Sahni et al. (1998).

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